

An algebraic study of convolution algebras*

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Abstract

We present a simple condition which guarantees a geometric convolution algebra to behave like a variant of the quasi-hereditary algebra. In particular, standard modules of the affine Hecke algebras of type BC , and the quiver Schur algebras are shown to satisfy the Brauer-Humphreys type reciprocity and the semi-orthogonality property. In addition, we present a new criterion of purity of weights in the geometric side. This yields a proof of Shoji's conjecture on limit symbols of type B [Shoji, Adv. Stud. Pure Math. 40 (2004)], and the purity of the exotic Springer fibers [K, Duke Math. 148 (2009)]. Using this, we describe the leading terms of the C^∞ -realization of a solution of the Lieb-McGuire system in the appendix. In [K, arXiv:1203.5254], we apply the results of this paper to the KLR algebras of type ADE to establish Kashwara's problem and Lusztig's conjecture.

Introduction

In representation theory of an algebra associated to a root datum, study of a geometric convolution algebra plays a major rôle. Introduced by Ginzburg [Gin85] in his study of affine Hecke algebras, it appeared in the study of the affine Hecke algebras [Lus88, Ari96, CG97, K09, VV11b], the BGG categories [Soe90, BGS96, ABG, Sch11], the Springer correspondence [CG97, Ach11, K11b, Rid12], the quantum loop algebras [Nak06], the Khovanov-Lauda-Rouquier algebras [Zhe08, Web10, VV11a], the quiver Schur algebras [VV99, SW11], and so on. Once appeared, it produces deep results in the spirit of the Kazhdan-Lusztig conjecture [BB81, BK81] and the Koszul-Langlands duality [BGS96, Soe00, ABG].

However, not much is known about the standard modules arising from this geometric convolution algebra construction. From the viewpoint of highest weight category [CPS88], standard modules should be indecomposable with simple heads, they should filter projective modules, and so on. These are not a part of the general theory of geometric convolution algebras [CG97] §8, and even its indecomposability is usually guaranteed by rather ad-hoc induction arguments.

*This is a revised and strengthened version of the first part of the paper “PBW bases and KLR algebras” arXiv:1203.5254. We divided it into two pieces since we learned that this part were invisible in the previous version of the above mentioned paper.

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Actually, several criteria which guarantee nice behavior of geometric convolution algebras are known [MV87, BGS96]. The problem is that many of the geometric convolution algebras arising from representation theory fail to satisfy such criteria. In particular, the resulting geometric convolution algebras rarely give rise to highest weight categories.

The goal of the present paper is to introduce two simple geometric conditions (\spadesuit) and (\clubsuit) which cover some algebras which are not covered by previous results, and to deduce their representation-theoretic consequences. Such an analysis, together with its algebraic interpretations, yields a proof of Shoji's conjecture in this paper. In addition, it serves as a basis of our proofs of Kashiwara's problem and Lusztig's conjecture in [K12b].

Let G be a connected algebraic group acting on a variety \mathfrak{X} over \mathbb{C} with finitely many orbits $\{\mathbb{O}_\lambda\}_{\lambda \in \Lambda}$ labeled by Λ . Let IC_λ be the minimal extension of the constant sheaf on \mathbb{O}_λ . We assume the following three conditions:

- (\spadesuit)₂ For each $\lambda \in \Lambda$, the G -orbit \mathbb{O}_λ has a connected stabilizer G_λ ;
- (\clubsuit)₁' For each $\lambda, \mu \in \Lambda$, the stalk of IC_λ along \mathbb{O}_μ satisfies the parity vanishing;
- (\clubsuit)₂ The natural map $H_G^\bullet(\mathrm{pt}) \rightarrow H_{G_\lambda}^\bullet(\mathrm{pt})$ is surjective for each $\lambda \in \Lambda$.

Here we replaced the condition (\clubsuit)₁ with a weaker condition (\clubsuit)₁' for the sake of simplicity. To such a pair (G, \mathfrak{X}) , we associate a convolution algebra

$$A = A_{(G, \mathfrak{X})} := \bigoplus_{\lambda, \mu} \mathrm{Ext}_{D_G^b(\mathfrak{X})}^\bullet(L_\lambda \boxtimes \mathrm{IC}_\lambda, L_\mu \boxtimes \mathrm{IC}_\mu),$$

where L_λ is a self-dual graded vector space for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, we have a simple graded A -module L_λ and its projective cover P_λ . Convolution algebras arising from the affine Hecke algebras of type A [CG97], type BC [K09], the KLR algebras of type ADE [VV11a, K12b], the quiver Schur algebras [Lu90a, VV99], and the algebra which governs the BGG category [Soe90, BGS96] satisfy our (actual) assumption. By refining the technique developed in [K11b], we prove:

Theorem A (= Theorem 3.5). *The global dimension of A is finite.*

This seems to be a new result for the quiver Schur algebras. For the KLR algebras, see [K12b]. For affine Hecke algebras, such a result is known by Opdam-Solleveld [OS09].

Since we deal with a variety \mathfrak{X} , we have a closure ordering among G -orbits that we denote by \prec . Let $A\text{-gmod}$ be the category of finitely generated graded A -modules. We define

$$\tilde{K}_\lambda := P_\lambda / \left(\sum_{\mu \prec \lambda, f \in \mathrm{hom}_A(P_\mu, P_\lambda)} \mathrm{Im} f \right) \quad \text{and} \quad K_\lambda := \tilde{K}_\lambda / \left(\sum_{f \in \mathrm{hom}_A(P_\lambda, \tilde{K}_\lambda) > 0} \mathrm{Im} f \right),$$

where hom denotes the graded homomorphism and $\mathrm{hom}(M, N)^j$ denotes the degree j -part of the homomorphism (for each $M, N \in A\text{-gmod}$).

Theorem B (= Theorems 1.3 & 1.6 + Proposition 3.8 + Corollary 3.11). *Under the above setting, we have:*

1. Each \tilde{K}_λ is a successive self-extension of K_λ . In addition, we have

$$[\tilde{K}_\lambda : L_\lambda] = \text{gdim } H_{\text{Stab}_G(x_\lambda)}^\bullet(\text{pt});$$

2. For each $\lambda, \mu \in \Lambda$, we have

$$\begin{aligned} \text{ext}_A^i(K_\lambda, K_\mu) &= \{0\} \quad \text{if } \lambda \not\succeq \mu \quad \text{and} \\ \text{ext}_A^i(\tilde{K}_\lambda, K_\mu^*) &= \begin{cases} \mathbb{C} & (\lambda = \mu, i = 0) \\ \{0\} & (\text{otherwise}) \end{cases}, \end{aligned}$$

where K_μ^* is the graded dual of K_μ regarded as an A -module;

3. As an equality of graded character expansion coefficients, we have

$$[P_\lambda : \tilde{K}_\mu] = [K_\mu : L_\lambda] \quad \text{for every } \lambda, \mu \in \Lambda;$$

4. We consider the positive characteristic analogue here. If the weight of A is pure, then the stalk of IC_λ along \mathbb{O}_μ is pure for every $\lambda, \mu \in \Lambda$;

5. If 4) holds, then each P_λ admits a filtration by $\{\tilde{K}_\mu\}_\mu$.

Note that the non-zero map in Theorem B 2) has its image L_λ since essentially the LHS is concentrated in degree ≥ 0 , and the RHS is concentrated in degree ≤ 0 . Also, Theorem B 1) implies the Cartan determinant formula of A (Corollary 3.12).

For a graded A -module M , we define

$$\text{gch } M := \sum_{\lambda \in \Lambda, k \in \mathbb{Z}} t^k [M : L_\lambda \langle k \rangle][\lambda],$$

where $[M : L_\lambda \langle k \rangle] \in \mathbb{Z}_{\geq 0}$ is the multiplicity of the grade k shift of L_λ in M .

Example C. Let $G = PGL(3, \mathbb{C})$ and let $\mathfrak{X} = \mathcal{N}$ be the nilpotent cone of \mathfrak{sl}_3 . Let W be the Weyl group of G and let \mathfrak{t} be the Cartan subalgebra of \mathfrak{sl}_3 . Then, we have $A = \mathbb{C}\mathfrak{S}_3 \ltimes \mathbb{C}[\mathfrak{t}]$ with $A^0 = \mathbb{C}\mathfrak{S}_3$ and $\deg \mathfrak{t}^* = 2$. We have $\Lambda = \{\text{triv} \succ \text{ref} \succ \text{sgn}\}$ and

$$\text{gch } K_{\text{triv}} = [\text{triv}], \text{gch } K_{\text{ref}} = [\text{ref}] + t^2[\text{triv}], \text{gch } K_{\text{sgn}} = [\text{sgn}] + (t^2 + t^4)[\text{ref}] + t^6[\text{triv}].$$

In addition, we have

$$H_{G_{\text{triv}}}^\bullet(\text{pt}) = \mathbb{C}, H_{G_{\text{ref}}}^\bullet(\text{pt}) = H_{\mathbb{C}^\times}^\bullet(\text{pt}) = \mathbb{C}[r], \text{ and } H_{G_{\text{sgn}}}^\bullet(\text{pt}) = \mathbb{C}[\mathfrak{t}]^{\mathfrak{S}_3},$$

where $\deg r = 2$. Theorem B 3) yields the following expression of the graded multiplicity matrix $[P : L] := ([P_\lambda : L_\gamma])_{\lambda, \gamma}$:

$$[P : L] = [P : \tilde{K}][\tilde{K} : K][K : L] = {}^t[K : L][\tilde{K} : K][K : L],$$

where $[P : \tilde{K}]$ and $[\tilde{K} : K]$ are graded multiplicity expansion coefficient matrices. This reads as:

$$\begin{aligned} & \frac{1}{(1-t^4)(1-t^6)} \begin{pmatrix} 1 & t^2+t^4 & t^6 \\ t^2+t^4 & 1+t^2+t^4+t^6 & t^2+t^4 \\ t^6 & t^2+t^4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ t^2 & 1 & 0 \\ t^6 & t^2+t^4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{1-t^2} & 0 \\ 0 & 0 & \frac{1}{(1-t^4)(1-t^6)} \end{pmatrix} \begin{pmatrix} 1 & t^2 & t^6 \\ 0 & 1 & t^2+t^4 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

In [Sho04] 3.13, Shoji conjectured that his limit Green function of type B gives the graded character of a coinvariant ring of a Weyl group of type B. Combining the results of Achar-Henderson [AH08] with Theorem B and our previous results, we prove:

Theorem D (Shoji's conjecture for type B = Theorem 5.3 + Corollary 5.6). *Let $G = Sp(2n, \mathbb{C})$ and let $\mathfrak{X} = \mathfrak{N}$ be its exotic nilpotent cone [K09]. Then, the modules K_λ arising from A are coinvariant algebras of type B. In particular, their graded characters are calculated by the Lusztig-Shoji algorithm. In addition, every exotic Springer fiber has a pure homology.*

Note that Theorem D also resolves [AH08] Conjecture 6.4.

As a bonus of Theorem D, we prove that the graded modules appearing from the Lieb-McGuire integrable systems [HO97] are exactly Shoji's coinvariant algebras in Appendix A.

The organization of this paper is as follows: In the first section, we formulate two conditions (\spadesuit) and (\clubsuit) and formulate our main results (Theorem 1.3, Theorem 1.5, and Theorem 1.6). In the second section, we make an induction using dg-modules to prove Theorem 1.3. In the third section, we prove Theorem A and other numerical consequences of Theorem 1.3 including the Brauer-Humphreys type reciprocity and the Cartan determinant formula. In the fourth section, we prove Theorem 1.5 and Theorem 1.6. Finally, we prove Shoji's conjecture in the fifth section. The appendix is devoted to the analysis of the Lieb-McGuire system.

Theorem B 3) shows that we have two versions of standard modules and they are deeply incorporated into the formulation. In addition, two versions of standard modules actually differ in Example C. Therefore, our result mainly points different direction from that of Cline-Parshall-Scott [CPS88]. Also, there is a notion of affine cellularity due to König-Xi [KX12], which provides a framework for algebraic results similar to ours. However, their algebraic conditions seem rather difficult to verify compared with our geometric conditions¹. Consequently, their results are applicable only for a small part of the algebras we concern in this paper or in [K12b] (at least at this moment).

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Convention

An algebra R is a (not necessarily commutative) unital \mathbb{C} -algebra. A variety \mathfrak{X} is a separated reduced scheme \mathfrak{X}_0 of finite type over some localization \mathbb{Z}_S of \mathbb{Z} specialized to \mathbb{C} . A G -variety is a variety with an action of a Chevalley group over \mathbb{Z}_S on \mathfrak{X}_0 specialized to \mathbb{C} . As in [BBD82] §6 and [BL94], we

¹For example, the combination of [K09] and §1 yields homological/categorical consequences parallel to the theory of affine cellular algebras for affine Hecke algebras of type BC with arbitrary rank and arbitrary real parameters (while the affine cellularity of affine Hecke algebras of type BC is known only in rank two case with generic real parameters [GM11]).

transplant the notion of weights to the derived category of (G -equivariant) constructible sheaves on \mathfrak{X} . Let us denote by $D^b(\mathfrak{X})$ (resp. $D^+(\mathfrak{X})$) the bounded (resp. bounded from the below) derived category of the category of constructible sheaves on \mathfrak{X} , and denote by $D_G^+(\mathfrak{X})$ the G -equivariant derived category of \mathfrak{X} . We have a natural forgetful functor $D_G^+(\mathfrak{X}) \rightarrow D^+(\mathfrak{X})$, whose preimage of $D^b(\mathfrak{X})$ is denoted by $D_G^b(\mathfrak{X})$. For an object of $D_G^b(\mathfrak{X})$, we may denote its image in $D^b(\mathfrak{X})$ by the same letter.

Let \mathbf{vec} be the category of \mathbb{Z} -graded vector spaces (over \mathbb{C}) bounded from the below so that its objects have finite-dimensional graded pieces. In particular, for $V = \oplus_{i \gg -\infty} V^i \in \mathbf{vec}$, its graded dimension $\mathbf{gdim} V := \sum_i t^i \dim V_i \in \mathbb{Z}((t))$ makes sense (with t being indeterminant). We define $V \langle m \rangle$ by setting $(V \langle m \rangle)_i := V_{i-m}$.

In this paper, a graded algebra A is always a \mathbb{C} -algebra whose underlying space is in \mathbf{vec} . Let $A\text{-gmod}$ be the category of finitely generated graded A -modules. For $E, F \in A\text{-gmod}$, we define $\mathrm{hom}_A(E, F)$ to be the direct sum of graded A -module homomorphisms $\mathrm{hom}_A(E, F)^j$ of degree j . We employ the same notation for extensions (i.e. $\mathrm{ext}_A^i(E, F) = \oplus_{j \in \mathbb{Z}} \mathrm{ext}_A^i(E, F)^j$). We denote by $\mathrm{lrr} A$ be the set of isomorphism classes of graded simple modules of A , and denote by $\mathrm{lrr}_0 A$ be the set of isomorphism classes of graded simple modules of A up to grading shifts. Two graded algebras are said to be Morita equivalent if their graded module categories are equivalent. For a graded A -module E , we denote its head by $\mathrm{hd} E$, and its socle by $\mathrm{soc} E$.

For $Q(t) \in \mathbb{Q}(t)$, we set $\overline{Q}(t) := Q(t^{-1})$. For derived functors $\mathbb{R}F$ or $\mathbb{L}F$ of some functor F , we represent its arbitrary graded piece (of its homology complex) by \mathbb{R}^*F or \mathbb{L}^*F , and the direct sum of whole graded pieces by $\mathbb{R}^\bullet F$ or $\mathbb{L}^\bullet F$. For example, $\mathbb{R}^*F \cong \mathbb{R}^*G$ means that $\mathbb{R}^i F \cong \mathbb{R}^i G$ for every $i \in \mathbb{Z}$, while $\mathbb{R}^\bullet F \cong \mathbb{R}^\bullet G$ means that $\bigoplus_i \mathbb{R}^i F \cong \bigoplus_i \mathbb{R}^i G$.

When working on some sort of derived category, we suppress \mathbb{R} or \mathbb{L} , or the category from the notation for simplicity when there is only small risk of confusion.

1 The conditions (\spadesuit), (\clubsuit) and main results

Let G be a connected reductive algebraic group. Let \mathfrak{X} be G -variety. Let Λ be the labelling set of G -orbits of \mathfrak{X} . For $\lambda \in \Lambda$, we denote the corresponding G -orbit by \mathbb{O}_λ . For $\lambda, \mu \in \Lambda$, we write $\lambda \preceq \mu$ if $\mathbb{O}_\lambda \subset \overline{\mathbb{O}_\mu}$. We assume the following property (\spadesuit):

(\spadesuit)₁ The set Λ is finite. For each $\lambda \in \Lambda$, we fix $x_\lambda \in \mathbb{O}_\lambda(\mathbb{C})$;

(\spadesuit)₂ For each $\lambda \in \Lambda$, the group $\mathrm{Stab}_G(x_\lambda)$ is connected.

We have a (relative) dualizing complex $\omega_{\mathfrak{X}} := p^! \mathbb{C} \in D_G^b(\mathfrak{X})$, where $p : \mathfrak{X} \rightarrow \{\mathrm{pt}\}$ is the G -equivariant structure map. We have a dualizing functor

$$\mathbb{D} : D_G^b(\mathfrak{X})^{op} \ni C^\bullet \mapsto \mathcal{H}om^\bullet(C^\bullet, \omega_{\mathfrak{X}}) \in D_G^b(\mathfrak{X}).$$

We have \mathbb{D} -autodual t -structure of $D_G^b(\mathfrak{X})$ whose truncation functor and perverse cohomology functor are denoted by τ and ${}^p H$, respectively. In particular, \mathbb{D} induces an equivalence of categories $\tau^{\geq 0} D_G^b(\mathfrak{X})^{op} \cong \tau^{\leq 0} D_G^b(\mathfrak{X})$ and each

$\mathcal{E} \in D_G^b(\mathfrak{X})$ admits a distinguished triangle

$$\tau^{<i}\mathcal{E} \rightarrow \mathcal{E} \rightarrow \tau^{\geq i}\mathcal{E} \xrightarrow{+1} \tau^{<i}\mathcal{E}[1]$$

for every $i \in \mathbb{Z}$.

For each $\lambda \in \Lambda$, we have a constant local system \mathbb{C}_λ on \mathbb{O}_λ . We have inclusions $i_\lambda : \{x_\lambda\} \hookrightarrow \mathfrak{X}$ and $j_\lambda : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$. Let $\mathbb{C}_\lambda := (j_\lambda)_! \mathbb{C}_\lambda[\dim \mathbb{O}_\lambda]$ and $\mathbb{IC}_\lambda := (j_\lambda)_! * \mathbb{C}_\lambda[\dim \mathbb{O}_\lambda]$ be the extension by zero and the minimal extension, which we regard as objects of $D_G^b(\mathfrak{X})$. We denote by

$$\begin{aligned} \text{Ext}_G^\bullet(\bullet, \bullet) : D_G^b(\mathfrak{X})^{op} \times D_G^b(\mathfrak{X}) &\longrightarrow D^+(\{\text{pt}\}) \\ \text{Ext}^\bullet(\bullet, \bullet) : D^b(\mathfrak{X})^{op} \times D^b(\mathfrak{X}) &\longrightarrow D^b(\{\text{pt}\}) \end{aligned}$$

the Ext (as bifunctors) of $D_G^b(\mathfrak{X})$ and $D^b(\mathfrak{X})$, respectively.

For each $\lambda \in \Lambda$, we fix $L_\lambda \in D^b(\text{pt})$ which is not quasi-isomorphic to $\{0\}$ and satisfying the self-duality condition $L_\lambda \cong L_\lambda^*$. We set

$$\mathcal{L} := \bigoplus_{\lambda \in \Lambda} L_\lambda \boxtimes \mathbb{IC}_\lambda \in D_G^b(\mathfrak{X}).$$

By construction, we find an isomorphism $\mathcal{L} \cong \mathbb{D}\mathcal{L}$.

We form a graded Yoneda algebra

$$A_{(G, \mathfrak{X})} = \bigoplus_{i \geq \mathbb{Z}} A_{(G, \mathfrak{X})}^i := \bigoplus_{i \in \mathbb{Z}} \text{Ext}_G^i(\mathcal{L}, \mathcal{L})$$

whose degree is the cohomological degree. We denote by $B_{(G, \mathfrak{X})}$ the algebra $A_{(G, \mathfrak{X})}$ by taking $\mathcal{L} = \bigoplus_{\lambda \in \Lambda} \mathbb{IC}_\lambda$ (and call it the basic ring of $A_{(G, \mathfrak{X})}$). The algebra $B_{(G, \mathfrak{X})}$ is Morita equivalent to $A_{(G, \mathfrak{X})}$, and hence all the arguments in the below are independent of the choice of \mathcal{L} , which we suppress for simplicity. We also drop (G, \mathfrak{X}) in case the meaning is clear from the context. We have $\text{Ext}^\bullet(\mathcal{L}, \mathcal{L}) \in D^b(\mathfrak{X})$, which implies $\dim \text{Ext}^\bullet(\mathcal{L}, \mathcal{L}) < \infty$. By the Serre spectral sequence

$$H_G^\bullet(\text{pt}) \otimes_{\mathbb{C}} \text{Ext}^\bullet(\mathcal{L}, \mathcal{L}) \Rightarrow \text{Ext}_G^\bullet(\mathcal{L}, \mathcal{L}) \cong A, \quad (1.1)$$

we conclude that $A \in \text{vec}$.

Lemma 1.1 (cf. Joshua [Jos97] §6). *The algebras A and B are isomorphic to their opposite rings A^{op} and B^{op} , respectively. In addition, the graded dual of L_λ is naturally isomorphic to L_λ as a graded A -module.*

Proof. The subalgebra

$$\bigoplus_{\lambda \in \Lambda} \text{end}_{\mathbb{C}}(L_\lambda) \boxtimes \text{Ext}_G^0(\mathbb{IC}_\lambda, \mathbb{IC}_\lambda) \subset A$$

is clearly symmetric and is the maximal semi-simple quotient of A . In addition, we have $L_\lambda^* \cong L_\lambda$ as a graded vector space. Therefore, it suffices to prove the corresponding statement for B by utilizing the non-degenerate pairing $L_\lambda \times L_\lambda \rightarrow \mathbb{C}$. Here the Verdier dualizing functor $\mathbb{D} : D_G^b(\mathfrak{X})^{op} \xrightarrow{\cong} D_G^b(\mathfrak{X})$ induces an isomorphism

$$\text{Ext}_G^*(\mathbb{IC}_\lambda, \mathbb{IC}_\mu) \cong \text{Ext}_G^*(\mathbb{D}\mathbb{IC}_\mu, \mathbb{D}\mathbb{IC}_\lambda) \cong \text{Ext}_G^*(\mathbb{IC}_\mu, \mathbb{IC}_\lambda).$$

Since this isomorphism is compatible with composition, we deduce that $B \cong B^{op}$. \square

Thanks to Lemma 1.1, for a graded A -module M , its graded dual M^* is again a graded A -module (which might not be finitely generated even M is).

For each $\lambda \in \Lambda$, we set

$$P_\lambda := \text{Ext}_G^\bullet(\mathbb{I}C_\lambda, \mathcal{L}) = \bigoplus_{i \in \mathbb{Z}} \text{Ext}_G^i(\mathbb{I}C_\lambda, \mathcal{L}).$$

Each P_λ is a graded projective left A -module. By construction, we have

$$A \cong \bigoplus_{\lambda \in \Lambda} L_\lambda^* \boxtimes \text{Ext}_G^\bullet(\mathbb{I}C_\lambda, \mathcal{L}) = \bigoplus_{\lambda \in \Lambda} L_\lambda^* \boxtimes P_\lambda$$

as left A -modules. It follows that P_λ is an indecomposable projective left A -module whose head is isomorphic to L_λ . We have an idempotent $e_\lambda \in A$ so that $P_\lambda \cong Ae_\lambda$ as left graded A -modules (up to a grading shift).

For each $\lambda \in \Lambda$, we set

$$\tilde{K}_\lambda := \text{Ext}_G^\bullet(C_\lambda, \mathcal{L}) \text{ and } K_\lambda := H^\bullet i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda].$$

We call K_λ a standard module, and \tilde{K}_λ a dual standard module of A . By adjunction, the Serre spectral sequence takes the form

$$E_2 = H_G^\bullet(\mathbb{O}_\lambda) \otimes_{\mathbb{C}} K_\lambda \cong H_{\text{Stab}_G(x_\lambda)}^\bullet(\{x_\lambda\}) \otimes_{\mathbb{C}} K_\lambda \Rightarrow \tilde{K}_\lambda. \quad (1.2)$$

We consider the following property (\clubsuit):

- (\clubsuit)₁ The spectral sequence (1.2) is E_2 -degenerate for each $\lambda \in \Lambda$;
- (\clubsuit)₂ The inclusion $\text{Stab}_G(x_\lambda) \subset G$ induces a surjection $H_G^\bullet(\text{pt}) \twoheadrightarrow H_{\text{Stab}_G(x_\lambda)}^\bullet(\text{pt})$.

Remark 1.2. Thanks to Lusztig [Lu90a] and Varagnolo-Vasserot [VV99], the quiver Schur algebras of type **A** also satisfy the properties (\spadesuit) and (\clubsuit) by taking \mathfrak{X} as the spaces of nilpotent representations of cyclic quivers with fixed dimension vectors.

For $M \in A\text{-gmod}$ and $i \in \mathbb{Z}$, we define

$$\begin{aligned} [M : L_\lambda \langle i \rangle]_0 &:= \dim \text{Hom}_A(P_\lambda \langle i \rangle, M) \in \mathbb{Z} \quad \text{and} \\ [M : L_\lambda] &:= \text{gdim } \text{hom}_A(P_\lambda, M) \in \mathbb{Z}((t)). \end{aligned}$$

We have $[M : L_\lambda] = \sum_{i \in \mathbb{Z}} [M : L_\lambda \langle i \rangle]_0 t^i \in \mathbb{Z}((t))$.

We define $A\text{-gmod}^{p^f}$ to be the full subcategory of $A\text{-gmod}$ consisting of objects that admit finite resolutions by finitely generated graded projective A -modules (this is an additive category). For $M \in A\text{-gmod}^{p^f}$ and $N \in A\text{-gmod}$, we define its graded Euler-Poincaré characteristic as:

$$\langle M, N \rangle_{\text{gEP}} := \sum_{i \geq 0} (-1)^i \text{gdim } \text{ext}_A^i(M, N) \in \mathbb{Z}((t)).$$

Theorem 1.3. *Assume the properties (\spadesuit) and (\clubsuit):*

1. *We have*

$$[\tilde{K}_\lambda : L_\mu] = 0 = [K_\lambda : L_\mu] \quad \text{for } \lambda \not\preceq \mu \quad \text{and} \quad [K_\lambda : L_\lambda] = 1;$$

2. For each $\mu \not\leq \lambda$, we have

$$\mathrm{ext}_A^\bullet(\tilde{K}_\lambda, \tilde{K}_\mu) = \{0\} \quad \text{and} \quad \mathrm{ext}_A^\bullet(K_\lambda, K_\mu) = \{0\};$$

3. For each $\lambda \in \Lambda$, we have

$$\tilde{K}_\lambda \cong P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right);$$

4. Each \tilde{K}_λ is a successive self-extension of K_λ . In addition, we have

$$[\tilde{K}_\lambda : L_\lambda] = \mathrm{gdim} H_{\mathrm{Stab}_G(x_\lambda)}^\bullet(\mathrm{pt}).$$

Remark 1.4. We can prove similar results if $\mathrm{Stab}_G(x_\lambda)$ is not connected. This is the situation we encounter in [K11b]. In fact, [K11b] and the semicontinuity of ext 's imply a variant of Theorem 1.3 in the framework of [Lus88]. Thus, Theorem 1.3 for affine Hecke algebras is new only in the framework of [K09].

In the following Theorem 1.5 and Theorem 1.6, we regard each IC_λ as a simple perverse sheaf of weight zero in the category of mixed sheaves on \mathfrak{X} via [BBD82] §5 and §6, and each L_λ as a mixed (complex of) vector space of weight zero. I.e. each L_λ^i is pure of weight i in the sense that the geometric Frobenius acts by $q^{i/2}\mathrm{id}$ if we switch the base field to the algebraic closure of a finite field of cardinality q . It follows that the algebra A and its standard modules $\{K_\lambda\}_{\lambda \in \Lambda}$ acquire (mixed) weight structures.

Theorem 1.5. *Assume the properties (\spadesuit) and (\clubsuit) . Then, each P_λ admits a (possibly infinite) separable A -module filtration so that its graded subquotients are $K_\mu \langle i \rangle$ with $\mu \preceq \lambda$ and $i \geq 0$. In addition, if all of IC_γ ($\gamma \in \Lambda$) are pointwise pure of weight 0 (see [BBD82] 5.1.8), then P_λ admits a finite separable A -module filtration so that its graded quotients are of the form $\tilde{K}_\mu \langle i \rangle$ with $\mu \preceq \lambda$ and $i \geq 0$.*

Theorem 1.6. *Assume the properties (\spadesuit) and (\clubsuit) . If the algebra A is pure in the sense A^j has weight j (for each $j \in \mathbb{Z}$), then each K_λ is pure. In particular, every IC_λ is pointwise pure of weight 0.*

Remark 1.7. The extra assumption of Theorem 1.6 holds for every KLR algebra [Lu90a], affine Hecke algebra of type BC with 2-parameters [K09], and quiver Schur algebra [Lu90a, VV99]. In particular, Theorem 1.6 presents new proofs of pointwise purity of the equivariant intersection cohomology complexes in the latter two cases (see Corollary 5.6).

The proofs of Theorem 1.3, Theorem 1.5, and Theorem 1.6 are given in section two and section four.

Corollary 1.8 (of Theorem 1.3). *The matrices $K := ([K_\lambda : L_\mu])$ and $\tilde{K} := ([\tilde{K}_\lambda : L_\mu])$ are upper-triangular and invertible in $\mathbb{Q}((t))$.*

Proof. Thanks to Theorem 1.3 1), K satisfies the desired property. Taking account into Theorem 1.3 4), we deduce that \tilde{K} is upper-triangular. Now the diagonal entries of \tilde{K} are invertible since they belong to $1 + t\mathbb{Z}[[t]]$ by Theorem 1.3 3). Therefore, we can invert the matrix \tilde{K} as desired. \square

2 DG-triangles and a proof of Theorem 1.3

Keep the setting of the previous section.

Lemma 2.1. *For each $\lambda \in \Lambda$, there exists a collection of objects $\{F^i \mathbb{C}_\lambda\}_{i \in \mathbb{Z}}$ of $D_G^b(\mathfrak{X})$ with the following properties:*

1. $F^k \mathbb{C}_\lambda = \mathbb{I} \mathbb{C}_\lambda$ for $k \geq 0$ and $F^k \mathbb{C}_\lambda = \mathbb{C}_\lambda$ for $k \ll 0$;
2. For each k , we have a distinguished triangle

$$\longrightarrow \mathbb{I} \mathbb{C}_{\lambda^k}[-d_k] \rightarrow F^k \mathbb{C}_\lambda \rightarrow F^{k+1} \mathbb{C}_\lambda \xrightarrow{+1} \quad \text{in } D_G^b(\mathfrak{X}),$$

where $\lambda^k \in \Lambda$ and d_k is a non-positive integer;

3. We have $\lambda^k \prec \lambda$ and $d_k \leq d_{k+1}$ for every k .

Proof. Since the embedding j_λ is locally closed, it follows that the functor $(j_\lambda)_!$ is right t -exact. For each m , the truncation $\tau^{\geq m} \mathbb{C}_\lambda$ satisfies the following distinguished triangle:

$$\longrightarrow {}^p H^m(\mathbb{C}_\lambda)[-m] \rightarrow \tau^{\geq m} \mathbb{C}_\lambda \rightarrow \tau^{\geq m+1} \mathbb{C}_\lambda \xrightarrow{+1} {}^p H^m(\mathbb{C}_\lambda)[-m+1].$$

In addition, ${}^p H^m(\mathbb{C}_\lambda)$ must be supported on $\overline{\mathbb{O}_\lambda}$ since all the constructions factor through the closed embedding $\overline{\mathbb{O}_\lambda} \subset \mathfrak{X}$.

Each ${}^p H^m(\mathbb{C}_\lambda)$ is a G -equivariant perverse sheaf, which is a successive extension of perverse sheaves on \mathfrak{X} obtained from constant local systems on G -orbits. Since the category of perverse sheaves on \mathfrak{X} (with respect to the stratification given by G -orbits) is artinian, ${}^p H^m(\mathbb{C}_\lambda)$ admits a Jordan-Hölder series

$$\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_\ell = {}^p H^m(\mathbb{C}_\lambda)$$

whose successive quotients are of the form $\mathbb{I} \mathbb{C}_{\lambda'}$ for some $\lambda' \prec \lambda$ except for $E_\ell/E_{\ell-1}$ in the $m = 0$ case. Applying the octahedron axiom to

$$\begin{array}{ccccc} & \tau^{\geq m} \mathbb{C}_\lambda & & E_i[-m] & \\ \swarrow & & \swarrow & & \swarrow +1 \\ \tau^{\geq m+1} \mathbb{C}_\lambda & \xrightarrow{+1} & {}^p H^m(\mathbb{C}_\lambda)[-m] & \longrightarrow & ({}^p H^m(\mathbb{C}_\lambda)/E_i)[-m] \end{array},$$

we obtain a complex \mathcal{E}_i^m with the following two distinguished triangles:

$$\begin{aligned} &\longrightarrow E_i[-m] \rightarrow \tau^{\geq m} \mathbb{C}_\lambda \rightarrow \mathcal{E}_i^m \xrightarrow{+1} E_i[-m+1] \\ &\longrightarrow ({}^p H^m(\mathbb{C}_\lambda)/E_i)[-m] \rightarrow \mathcal{E}_i^m \rightarrow \tau^{\geq m+1} \mathbb{C}_\lambda \xrightarrow{+1} ({}^p H^m(\mathbb{C}_\lambda)/E_i)[-m+1]. \end{aligned}$$

Notice that the canonical map $\tau^{\geq m} \mathbb{C}_\lambda \rightarrow \tau^{\geq m+1} \mathbb{C}_\lambda$ factors through \mathcal{E}_i^m . Hence we obtain a distinguished triangle

$$\longrightarrow (E_{i+1}/E_i)[-m] \rightarrow \mathcal{E}_i^m \rightarrow \mathcal{E}_{i+1}^m \xrightarrow{+1} (E_{i+1}/E_i)[-m+1]$$

for each i .

Therefore, by arranging the sequence \mathcal{E}_i^m with respect to the lexicographic order of (m, i) , we obtain a collection of objects $F^i \mathbb{C}_\lambda$ with $d_i = -m$ which satisfies the assertions except for $F^i \mathbb{C}_\lambda = \mathbb{C}_\lambda$ for $i \ll 0$.

Each of $\tau^{\geq m}\mathbb{C}_\lambda$ is an object of $D_G^b(\mathfrak{X})$. Therefore, we deduce that each \mathcal{E}_i^m belongs to $D_G^b(\mathfrak{X})$. Applying the forgetful map, we can regard \mathcal{E}_i^m as a bounded complex of sheaves cohomologically constructible with respect to the G -orbits. Since the bounded derived category of the category of perverse sheaves is equal to $D^b(\mathfrak{X})$, we conclude that ${}^pH^m(\mathbb{C}_\lambda) = \{0\}$ for $m \ll 0$. This, together with the fact $\ell < \infty$, implies $F^i\mathbb{C}_\lambda = \mathbb{C}_\lambda$ for $i \ll 0$ as required. \square

For each k , we set

$$P_\lambda^{(k)} := \text{Ext}_G^\bullet(F^k\mathbb{C}_\lambda, \mathcal{L}).$$

This is a collection of left A -modules so that $P_\lambda^{(0)} = P_\lambda$ and $P_\lambda^{(k)} = K_\lambda$ for $k \ll 0$. By adding a trivial differential d of degree one on $A = \oplus_i A^i$, we regard A as a DG-algebra (DG stands for differential graded). We sometimes regard $P_\lambda^{(k)}$ as a left A -dgmodule with a trivial differential so that its grading coincides with its degree in the complex.

Each P_λ is a \mathcal{K} -projective A -dgmodule (with trivial differential) since it is projective as an A -module ([BL94] 10.12.1). Let \mathcal{D}_A denote the derived category of the category of A -dgmodules ([BL94] 10.4.1).

For each k , we have the following distinguished triangle of A -dgmodules:

$$\rightarrow P_\lambda^{(k+1)} \rightarrow P_\lambda^{(k)} \rightarrow P_{\lambda^k}[d_k] \xrightarrow{+1}.$$

Note that since we deal with dgmodules, the shift functor is the same as the degree shift as A -modules (up to sign twists of differentials).

Lemma 2.2. *The algebra A is left and right Noetherian.*

Proof. Let C be the image of the natural map $H_G^\bullet(\text{pt}) \rightarrow A$. Since $H_G^\bullet(\text{pt})$ is a polynomial algebra, we deduce that C is a Noetherian algebra. By (1.1), A is a finitely generated module over C . Hence, every C -submodule of A is again finitely generated. This particularly applies to every left or right A -submodule of A , and hence the result. \square

Lemma 2.3. *Every finitely generated left A -dgmodule M with a trivial differential d_M admits a quasi-isomorphism from a finitely generated A -dgmodule $(B(M), d_{B(M)})$ with the following properties:*

1. *We have an isomorphism*

$$B(M) \cong \bigoplus_{i \geq 0} B_i(M)$$

as graded A -modules, where we have

$$B_i(M) \cong \bigoplus_{\lambda \in \Lambda, j \in \mathbb{Z}} P_\lambda[j]^{\oplus m_{\lambda,j}^i} \quad \text{as graded } A\text{-modules};$$

2. *For each $i \geq 0$, we have $d_{B(M)}(B_i(M)) \subset B_{i-1}(M)$ with $B_{-1}(M) := \{0\}$;*
3. *For each $k \in \mathbb{Z}$, the twisted degree k -part*

$$B^\sharp(M)^k := \bigoplus_{j \geq 1} B_j(M)^{k-j} \subset B(M)$$

is finite-dimensional.

Proof. We have a graded projective A -resolution

$$\cdots \longrightarrow P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow M \rightarrow 0$$

constructed so that P_j is the minimal projective module which surjects onto $\ker(P_{j-1} \rightarrow P_{j-2})$ (or $\ker(P_0 \rightarrow M)$ if $j = 1$). As M is finitely generated as a left A -module, we deduce that P_0 is finitely generated. Since the kernel of a map between finitely generated modules of a Noetherian algebra is finitely generated, we conclude that each P_j is finitely generated by induction on j . In particular, each graded piece of P_j is finite dimensional.

Since M is finitely generated as a left A -module, there exists $n_0 \in \mathbb{Z}$ so that $M^k = 0$ for every $k < n_0$. It follows that $P_j^{n_0+j-i} = \{0\}$ for $i > 0$. Therefore, the direct sum $\bigoplus_j P_j[j]$ satisfies the third assertion.

By setting $B(M) := \bigoplus_{j \geq 0} P_j[j]$ and

$$d_{B(M)} = (-1)^j d_j : P_j[j] \longrightarrow P_{j-1}[j-1],$$

we deduce an A -dgmodule which is quasi-isomorphic to M (and satisfies the requirement of the first two assertions) by setting $f : B(M) \rightarrow M$ to be $f(P_j[j]) = \{0\}$ unless $j = 0$ and $f|_{P_0} = d_0$. \square

Theorem 2.4 (Bernstein-Lunts [BL94] 11.3.1). *In the setting of Lemma 2.3, let N be also a finitely generated A -dgmodule with a trivial differential d_N . Then, we have*

$$H^k(\mathbb{R}\mathrm{Hom}_{\mathcal{D}_A}(M, N)) \cong \bigoplus_{k=j-i} \mathrm{ext}_A^i(M, N)^j.$$

Proof. The original assertion [BL94] 11.3.1 is for $A \cong H_G^\bullet(\{\mathrm{pt}\})$, but it carries over this case. The degree shift arises since the i -th term of a projective resolution has degree shift i (a differential in an A -dgmodule has degree one, while a differential in a complex of graded A -modules has degree zero). \square

In this paper, we call an A -dgmodule (P, d_P) projective if P is isomorphic to a projective (graded) A -module. A projective A -dgmodule (P, d_P) is finitely generated if P is a direct sum of finitely many indecomposable projective modules as (graded) A -modules.

Proposition 2.5. *An A -dgmodule (M, d_M) with a trivial differential admits a quasi-isomorphism from a finitely generated projective A -dgmodule M^+ if and only if there exists a finite interval $I = [a, b]$ so that*

$$\mathrm{ext}_A^i(M, L_\lambda) = \{0\} \text{ for every } i \notin [a, b] \text{ and } \lambda \in \Lambda. \quad (2.1)$$

In this case, if we additionally have

$$\dim \mathrm{ext}_A^\bullet(M, L_\lambda) = n$$

for some λ , then we can arrange M^+ if necessary so that $\{P_\lambda[k]\}_{k \in \mathbb{Z}}$ appears exactly n -times in M^+ as direct factors of a graded A -module.

Proof. We prove the first part of the assertion.

Assume that M admits a quasi-isomorphism from $M^+ = (M^+, d_{M^+})$, where M^+ is a finitely generated projective A -dgmodule. By assumption, we deduce that the complex $\bigoplus_k \text{Hom}_A(M^+, \bigoplus_\lambda L_\lambda[k])$ is bounded and has finite-dimensional graded pieces. Therefore, we deduce the assertion (\Rightarrow by Theorem 2.4 and the fact that the homology of a bounded finite-dimensional complex is bounded and finite-dimensional).

We prove the reverse implication. By Lemma 2.3, we deduce that

$$\dim \text{ext}_A^i(M, L_\lambda) < \infty$$

for each i and $\lambda \in \Lambda$. By assumption, we deduce

$$\bigoplus_{\lambda \in \Lambda, i \in \mathbb{Z}} \dim \text{ext}_A^i(M, L_\lambda) < \infty.$$

Since the projective A -dgmodule $B(M)$ borrowed from Lemma 2.3 can be constructed from a minimal graded projective A -resolution, we can rearrange it if necessary to assume that $B(M)$ is finitely generated. This proves the (\Leftarrow)-part of the first assertion.

For the second assertion, we only need to notice the procedures of the construction of M^+ only assigns a single copy of P_λ to a one-dimensional ext. \square

Corollary 2.6. *A graded A -module M belongs to $A\text{-gmod}^{pf}$ if and only if M admits a quasi-isomorphism from a finitely generated projective A -dgmodule. \square*

Corollary 2.7. *Assume that $P_\lambda^{(k)}$ admits a quasi-isomorphism from a finitely generated projective A -dgmodule. Then, $P_\lambda^{(k-1)}$ also admits a quasi-isomorphism from a finitely generated projective A -dgmodule.*

Proof. Straight-forward by Proposition 2.5 and the fact that $\mathbb{R}\text{Hom}_{\mathcal{D}_A}$ is a triangulated functor. \square

Corollary 2.8 (of Corollary 2.7). *Let $\lambda \in \Lambda$ and regard \tilde{K}_λ as an A -dgmodule with a trivial differential. We have:*

1. $\tilde{K}_\lambda \in A\text{-gmod}^{pf}$;
2. *there exists a projective A -dgmodule M^+ so that: 1) M^+ admits a quasi-isomorphism to \tilde{K}_λ , and 2) M^+ is a direct sum of one copy of P_λ and finitely many copies of $\{P_\mu[k]\}_{\mu \prec \lambda, k \in \mathbb{Z}}$ as a graded A -module.*

Proof. The first assertion follows from Corollary 2.7. For the second assertion, we apply the second part of Proposition 2.5 to the construction of $P_\lambda^{(k)}$. \square

Proof of Theorem 1.3. For $\mu \not\prec \lambda$, we have $j_\lambda^! \mathbb{IC}_\mu = \{0\}$ by the support condition. It follows that

$$[\tilde{K}_\lambda : L_\mu \langle i \rangle]_0 = \dim \text{Ext}_{D_G^b(\mathfrak{X})}^i(\mathbb{C}_\lambda, \mathbb{IC}_\mu) = \dim \text{Ext}_{D_G^b(\mathbb{O}_\lambda)}^{i - \dim \mathbb{O}_\lambda}(\underline{\mathbb{C}}_\lambda, j_\lambda^! \mathbb{IC}_\mu) = 0, \quad (2.2)$$

$$[K_\lambda : L_\mu \langle i \rangle]_0 = \dim H^{i + \dim \mathbb{O}_\lambda} i_\lambda^! \mathbb{IC}_\mu = 0 \quad (2.3)$$

for every $i \in \mathbb{Z}$. In addition, we have

$$[K_\lambda : L_\lambda \langle i \rangle]_0 = \dim H^{i+\dim \mathbb{O}_\lambda} i_\lambda^! \mathbf{IC}_\lambda = \begin{cases} 1 & (i = 0) \\ 0 & (i \neq 0) \end{cases}. \quad (2.4)$$

This proves Theorem 1.3 1).

By Corollary 2.8 and (2.2), it is straight-forward to see

$$\mathrm{ext}_A^*(\tilde{K}_\lambda, \tilde{K}_\mu) = \{0\} \quad (2.5)$$

unless $\mu \preceq \lambda$. This is the first part of Theorem 1.3 2).

This, together with (2.2) implies

$$\mathrm{hom}_A(\tilde{K}_\mu, L_\lambda) = \{0\} \quad (2.6)$$

except for the case $\lambda = \mu$.

Therefore, we deduce a well-defined map

$$\psi_\lambda : P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right) \longrightarrow \tilde{K}_\lambda.$$

Let $\ker := \ker \psi_\lambda$ and $\mathrm{coker} := \mathrm{coker} \psi_\lambda$. Each simple quotient of \ker is of the form $L_\gamma \langle i \rangle$ with $\gamma \not\preceq \lambda$ and $i > 0$. If $\ker \neq \{0\}$, then the long exact sequence shows

$$\mathrm{ext}_A^1(\tilde{K}_\lambda, L_\gamma) \neq \{0\},$$

which contradicts with Corollary 2.8 2). Therefore, we conclude that $\mathrm{hd} \ker = \{0\}$. Each simple quotient of coker is of the form $L_\gamma \langle k \rangle$ with $\gamma \succeq \lambda$ and $k > 0$. By (2.6), the head of coker is isomorphic to a direct sum of $\{L_\lambda \langle k \rangle\}_{k \in \mathbb{Z}}$. We have

$$[\tilde{K}_\lambda : L_\lambda \langle i \rangle]_0 = \dim \mathrm{Ext}_{D_G^b(\mathbb{O}_\lambda)}^{i-\dim \mathbb{O}_\lambda}(\mathbb{C}_\lambda, j_\lambda^! \mathbf{IC}_\lambda) = \dim H_{\mathrm{Stab}_G(x_\lambda)}^i(\mathrm{pt}). \quad (2.7)$$

This proves the latter half of Theorem 1.3 4).

Thanks to $(\clubsuit)_2$, we have the following map

$$\begin{aligned} H_G^\bullet(\mathrm{pt}) &\twoheadrightarrow H_{\mathrm{Stab}_G(x_\lambda)}^\bullet(\mathrm{pt}) \cong \mathrm{Ext}_G^\bullet(\mathbb{C}_\lambda[\dim \mathbb{O}_\lambda], j_\lambda^! \mathbf{IC}_\lambda) \\ &\cong \mathrm{Ext}_G^\bullet((j_\lambda)_! \mathbb{C}_\lambda[\dim \mathbb{O}_\lambda], \mathbf{IC}_\lambda) \subset \tilde{K}_\lambda. \end{aligned} \quad (2.8)$$

Since $H_G^\bullet(\mathrm{pt})$ is the multiplication of cohomology class of the base space BG , we deduce that $H_G^\bullet(\mathrm{pt})$ maps to the center of A . The above map (2.8) implies that every graded composition factor of \tilde{K}_λ isomorphic to $L_\lambda \langle i \rangle$ ($i > 0$) is reached from L_λ by applying the action of the center of A . Therefore, ψ_λ must be surjective, and hence it is an isomorphism. Therefore, we conclude Theorem 1.3 3).

For each $f \in \mathrm{hom}_A(\tilde{K}_\lambda^{\oplus N_0}, \tilde{K}_\lambda)_d$ with some $N_0, d > 0$, we see that

$$\begin{aligned} \mathrm{ext}_A^*(\ker f \oplus \mathrm{coker} f[-1], \tilde{K}_\gamma) &\neq \{0\}, \\ \mathrm{ext}_A^*(\ker f \oplus \mathrm{coker} f[-1], L_\gamma) &\neq \{0\}, \text{ or} \\ \mathrm{ext}_A^*(\tilde{K}_\delta, \ker f \oplus \mathrm{coker} f[-1]) &\neq \{0\} \end{aligned} \quad (2.9)$$

only if $\gamma \preceq \lambda$ or $\delta \succeq \lambda$ by the long exact sequences applied to (2.5). It follows that every simple quotient of $\operatorname{coker} f$ or $\ker f$ is isomorphic to $L_\lambda \langle k \rangle$ for some k . Therefore, we conclude that $\operatorname{coker} f$ admits a surjection from $\tilde{K}_\lambda^{\oplus N_1}$ for some $N_1 > 0$ as A -modules. In addition, we have

$$\operatorname{ext}_A^*(\operatorname{coker} f, \tilde{K}_\gamma) \neq \{0\}, \text{ or } \operatorname{ext}_A^*(\tilde{K}_\delta, \operatorname{coker} f) \neq \{0\} \quad (2.10)$$

only if $\gamma \preceq \lambda$ or $\delta \succeq \lambda$, respectively.

By the assumption (\clubsuit) , we deduce that

$$H_G^{>k}(\mathbb{O}_\lambda) \otimes_{\mathbb{C}} K_\lambda \subset \tilde{K}_\lambda$$

is an A -submodule for each k . Therefore, \tilde{K}_λ admits a decreasing filtration by A -submodules whose graded quotients are direct sums of A -modules whose graded composition multiplicity is the same as K_λ up to uniform shifts. In addition, this exhibits that K_λ is a quotient of \tilde{K}_λ . Hence K_λ must be indecomposable and it is generated by a single copy of L_λ sitting at degree 0.

Here the action of $H_G^\bullet(\mathbb{O}_\lambda)$ is achieved by the center of A , which commutes with all the multiplications by A . It follows that \tilde{K}_λ is a successive extension of K_λ , which is the first part of Theorem 1.3 4).

By the multiplicity estimate (2.4), we conclude that the natural surjection $\tilde{K}_\lambda \rightarrow K_\lambda$ obtained by forgetting the equivariant structure, is of the form

$$V \otimes_{\mathbb{C}} P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right) \longrightarrow P_\lambda / \left(\sum_{\mu \prec \lambda} A e_\mu P_\lambda \right) \longrightarrow K_\lambda \rightarrow 0 \quad (\text{exact}),$$

where $V \in \mathbf{vec}$ is the multiplicity-space.

In particular, (2.10) implies that

$$\operatorname{ext}_A^*(K_\lambda, \tilde{K}_\gamma) \neq \{0\}, \text{ or } \operatorname{ext}_A^*(\tilde{K}_\delta, K_\lambda) \neq \{0\}$$

only if $\gamma \preceq \lambda$ or $\delta \succeq \lambda$, respectively. By applying similar argument as above to \tilde{K}_γ or \tilde{K}_δ , we conclude the second half of Theorem 1.3 2), which completes the proof of Theorem 1.3. \square

3 Applications of Theorem 1.3

Keep the setting of the previous section.

Lemma 3.1. *For each $\lambda \in \Lambda$, we have $\dim K_\lambda < \infty$.*

Proof. The assertion is equivalent to $\dim H^\bullet i_\lambda^! \mathcal{L} < \infty$, which follows from $i_\lambda^! : D^b(\mathfrak{X}) \rightarrow D^b(\{x_\lambda\})$ and $\mathcal{L} \in D^b(\mathfrak{X})$. \square

Corollary 3.2. *For each $\lambda \in \Lambda$, the A -module K_λ has a finite composition series.* \square

Corollary 3.3. *For each $\lambda \in \Lambda$, the A -module K_λ is quasi-isomorphic to a finitely generated projective A -dgmodule.*

Proof. By Theorem 1.3 3) and 4), we have $\operatorname{end}_A(\tilde{K}_\lambda) \cong H_{\operatorname{Stab}_G(x_\lambda)}^\bullet(\operatorname{pt})$. The latter is a polynomial ring. Therefore, K_λ admits a (finite length) Koszul type resolution by \tilde{K}_λ . Therefore, Corollary 2.8 implies the result. \square

Corollary 3.4. *For each $\lambda \in \Lambda$, the A -module L_λ is quasi-isomorphic to a finitely generated projective A -dgmodule. In particular, L_λ admits a projective resolution of finite length.*

Proof. For the first assertion, combine Corollary 3.3, Corollary 3.2, Theorem 1.3 1), and the fact that $\mathbb{R}\mathrm{Hom}_{\mathcal{D}_A}$ is a triangulated functor. The second assertion follows by Corollary 2.6. \square

Theorem 3.5. *Assume the properties (\spadesuit) and (\clubsuit) . Then, the algebra A has finite global dimension. In other words, we have $A\text{-gmod}^{pf} \xrightarrow{\cong} A\text{-gmod}$.*

Proof. The graded algebra A is (left and right) Noether, and is Morita equivalent to B . The graded algebra B is non-negatively graded and B^0 is (canonically isomorphic to) the semisimple quotient of the Jacobson radical $B^{>0}$ of B . Therefore, we apply Li's result [Li96] to Corollary 3.4 to conclude the result. \square

For each $M \in A\text{-gmod}$, we define its graded character as:

$$\mathrm{gch} M := \sum_{\lambda \in \Lambda} [M : L_\lambda] [L_\lambda] \in \bigoplus_{\lambda \in \Lambda} \mathbb{Z}((t)) [L_\lambda].$$

Lemma 3.6. *Each of the collections $\{\mathrm{gch} K_\lambda\}_{\lambda \in \Lambda}$ and $\{\mathrm{gch} \tilde{K}_\lambda\}_{\lambda \in \Lambda}$ is a $\mathbb{Z}((t))$ -basis of $\bigoplus_{\lambda \in \Lambda} \mathbb{Z}((t)) [L_\lambda]$.*

Proof. The first assertion is a direct consequence of Theorem 1.3 1) and 4). For the second assertion, we use the fact that $\mathrm{gch} P_\lambda \equiv [L_\lambda]$ modulo $t\mathbb{Z}[[t]]$ (and each element of $1 + t\mathbb{Z}[[t]]$ is invertible). \square

For each $M \in A\text{-gmod}$, we have collections of elements $[M : P_\lambda]$, $[M : K_\lambda]$ and $[M : \tilde{K}_\lambda]$ in $\mathbb{Q}((t))$ so that

$$\mathrm{gch} M = \sum_{\lambda \in \Lambda} [M : P_\lambda] \mathrm{gch} P_\lambda = \sum_{\lambda \in \Lambda} [M : K_\lambda] \mathrm{gch} K_\lambda = \sum_{\lambda \in \Lambda} [M : \tilde{K}_\lambda] \mathrm{gch} \tilde{K}_\lambda.$$

Thanks to Theorem 3.5, every module in $A\text{-gmod}$ can be a variable of the graded Euler-Poincaré pairing.

Lemma 3.7. *If $M \in A\text{-gmod}$ is finite-dimensional, then its graded dual M^* belongs to $A\text{-gmod}$. In addition, we have*

$$[M : L_\mu] = \overline{[M^* : L_\mu]}.$$

Proof. See Lemma 1.1 and [K11b] Lemma 2.5. \square

Proposition 3.8. *We have*

$$\mathrm{ext}_A^i(\tilde{K}_\lambda, K_\mu^*) \cong \begin{cases} \mathbb{C} & (\lambda = \mu, i = 0) \\ \{0\} & (\text{otherwise}) \end{cases}, \text{ and } \left\langle \tilde{K}_\lambda, K_\mu^* \right\rangle_{\mathrm{gEP}} = \begin{cases} 1 & (\lambda = \mu) \\ 0 & (\lambda \neq \mu) \end{cases}.$$

Remark 3.9. Proposition 3.8 can be seen as the “dual picture” of Mirolli-Vilonen and Beilinson-Ginzburg-Soergel [MV87, BGS96], or the equivariant picture of Chriss-Ginzburg [CG97] §8.7.

Proof of Proposition 3.8. By Corollary 2.8 2) and Theorem 1.3 1), we deduce the case $\mu \not\prec \lambda$.

In the case of $\mu \prec \lambda$, we derive

$$\mathrm{hom}_A(M, N) \cong \mathrm{hom}_A(N^*, M^*)$$

to deduce that

$$\mathrm{ext}_A^\bullet(K_\mu, K_\lambda^*) = \{0\}, \text{ and } \langle K_\mu, K_\lambda^* \rangle_{\mathrm{gEP}} = 0 \quad (3.1)$$

from Corollary 2.8. We have a separable decreasing filtration $\{F^k \tilde{K}_\lambda\}_{k \geq 0}$ so that its associated graded is a direct sum of grading shifts of K_λ by Theorem 1.3 4). We have

$$\varinjlim \mathrm{ext}_A^\bullet(K_\mu, (F^k \tilde{K}_\lambda)^*) \cong \mathrm{ext}_A^\bullet(K_\mu, \varinjlim (F^k \tilde{K}_\lambda)^*) = \{0\}.$$

Therefore, (3.1) is equivalent to the (remaining part of the) assertion. \square

Corollary 3.10. *We have*

$$\langle K_\lambda, K_\mu^* \rangle_{\mathrm{gEP}} = 0 \quad (\lambda \neq \mu).$$

Proof. The anti-linearity of $\langle E, F \rangle_{\mathrm{gEP}}$ with respect to E and Theorem 1.3 4) implies the result (cf. (3.1)). \square

Corollary 3.11. *Assume the properties (\spadesuit) and (\clubsuit) . Then, we have*

$$[P_\lambda : \tilde{K}_\mu] = [K_\mu : L_\lambda] \quad \text{for each } \lambda, \mu \in \Lambda.$$

Proof. We have

$$\begin{aligned} \delta_{\lambda, \mu} &= \langle P_\lambda, L_\mu^* \rangle_{\mathrm{gEP}} = \sum_{\gamma, \delta} \overline{[P_\lambda : \tilde{K}_\gamma]} \overline{[L_\mu : K_\delta]} \langle \tilde{K}_\gamma, K_\delta^* \rangle_{\mathrm{gEP}} \\ &= \sum_{\gamma} \overline{[P_\lambda : \tilde{K}_\gamma]} \overline{[L_\mu : K_\gamma]} \end{aligned}$$

by Proposition 3.8. By applying the bar involution, this shows

$$([P_\lambda : \tilde{K}_\gamma])([K_\delta : L_\mu])^{-1} = (\delta_{\lambda, \mu}),$$

which is equivalent to the assertion. \square

Corollary 3.12 (Cartan determinant formula). *We set*

$$[P : L] := ([P_\lambda : L_\mu])_{\lambda, \mu \in \Lambda} = (\langle P_\mu, P_\lambda \rangle_{\mathrm{gEP}})_{\lambda, \mu \in \Lambda}$$

as the square matrix of size $\#\Lambda$. We have

$$\det [P : L] = \prod_{\lambda \in \Lambda} \mathrm{gdim} H_{\mathrm{Stab}_G(x_\lambda)}^\bullet(\{\mathrm{pt}\}).$$

Proof. As in the proof of Corollary 3.11, we factorize

$$[P : L] = [P : \tilde{K}][\tilde{K} : K][K : L],$$

where the matrices in the RHS are the $\#\Lambda$ -square matrices consisting of the expansion coefficients between projectives/dual standards, dual standards/standards, and standards/simples. By Theorem 1.3 1), the determinant of the third matrix is 1. By Theorem 3.11, the determinant of the first matrix is also 1. Hence, the result follows from Theorem 1.3 4). \square

4 Proofs of Theorem 1.5 and Theorem 1.6

Keep the setting of the previous section.

Lemma 4.1. *Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G -stable subvariety. Then, \mathfrak{Y} satisfies the conditions (\spadesuit) and (\clubsuit) if \mathfrak{X} does.*

Proof. The condition $(\clubsuit)_1$ is satisfied since all the restrictions to orbits (in \mathfrak{Y}) factor through j . Since the other conditions become weaker by the restriction to any G -invariant subset, we conclude the result. \square

Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G -stable subvariety. We form a graded algebra

$$A_{(G, \mathfrak{Y})} := \text{Ext}_G^\bullet(j^* \mathcal{L}, j^* \mathcal{L}).$$

Proposition 4.2. *Let $i : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$ be the inclusion of a closed G -orbit (with $\lambda \in \Lambda$), and let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be its complement. Then, $A_{(G, \mathfrak{Y})}$ is the quotient of $A_{(G, \mathfrak{X})}$ obtained by annihilating all simple $A_{(G, \mathfrak{X})}$ -module constituents isomorphic to L_λ . In addition, its kernel admits a finite length graded A -module resolution by $\{\tilde{K}_\lambda \langle k \rangle\}_{k \in \mathbb{Z}}$.*

Proof. By Morita equivalences, we switch to the basic rings and denote them by $A' := B_{(G, \mathfrak{Y})}$ and $A := B_{(G, \mathfrak{X})}$. In particular, we assume that the both A and A' are non-negatively graded rings.

The algebra map $A \rightarrow A'$ is the restriction map

$$\text{Ext}_G^*(\mathcal{L}, \mathcal{L}) \longrightarrow \text{Ext}_G^*(j^* \mathcal{L}, j^* \mathcal{L}) \cong \text{Ext}_G^*(j^! \mathcal{L}, j^! \mathcal{L}),$$

where we used $j^* \cong j^!$ for an open embedding j . It follows that this map, viewed as a left A -module, is given by

$$\text{Ext}_G^*(\mathcal{L}, \mathcal{L}) \longrightarrow \text{Ext}_G^*(j_! j^! \mathcal{L}, \mathcal{L}).$$

We have a distinguished triangle

$$\rightarrow j_! j^! \mathcal{L} \rightarrow \mathcal{L} \rightarrow i_* i^* \mathcal{L} \xrightarrow{+1}.$$

Let us write the resulting distinguished triangle as left A -dgmodules as:

$$\rightarrow K \rightarrow A \rightarrow A' \xrightarrow{+1}.$$

The sheaf $i_* i^* \mathcal{L}$ is presented as a finitely many successive distinguished triangles between $\{\mathbb{C}_\lambda[k]\}_{k \in \mathbb{Z}}$ since it is a constructible sheaf with respect to the closed G -orbit \mathbb{O}_λ . It follows that K is presented as a finitely many successive distinguished triangles between $\{\tilde{K}_\lambda[-k]\}_{k \in \mathbb{Z}}$ as an R_β -dgmodule. By Proposition 2.5 and Corollary 2.8, we deduce that K admits a quasi-isomorphism from a finitely-generated projective A -dgmodule whose direct summands are $\{P_\lambda[-k]\}_{k \in \mathbb{Z}}$. In conjunction with Theorem 2.4, each $\lambda \neq \mu \in \Lambda$ satisfies

$$\text{ext}_A^\bullet(A, L_\mu) \cong \text{ext}_A^\bullet(A', L_\mu) \quad (4.1)$$

as ungraded vector spaces.

Thanks to the projectivity of A and Theorem 2.4, we conclude that

$$\text{ext}_A^i(A, L_\mu)^j = \{0\} \quad \text{unless} \quad i = 0 = j.$$

By the construction of A and A' , it follows that the map $A \rightarrow A'$ gives rise to an isomorphism between their simple quotients isomorphic to L_μ (with $\lambda \neq \mu \in \Lambda$). It follows that

$$\mathrm{ext}_A^i(A', L_\mu)^j = \{0\} \quad \text{unless} \quad i = 0 = j,$$

by dimension counting, and we have

$$\mathrm{ext}_A^*(A, L_\mu) \cong \mathrm{ext}_A^*(A', L_\mu). \quad (4.2)$$

In addition, we have $[A' : L_\lambda]_A = 0$ since L_λ is the multiplicity space of IC_λ in \mathcal{L} and $j^! \mathrm{IC}_\lambda = \{0\}$. This implies that A' is obtained by annihilating all the L_λ in its projective cover as left A -modules.

For each $\lambda \neq \mu \in \Lambda$, we set

$$P'_\mu := P_\mu / \left(\sum_{f \in \mathrm{hom}_A(P_\lambda, P_\mu)} \mathrm{Im} f \right).$$

The above analysis implies that A' is a direct sum of $\{P'_\mu \langle k \rangle\}_{\mu \neq \lambda, k \in \mathbb{Z}}$ as a left A -module.

The map $A \rightarrow A'$ gives rise to a surjection between quotients by their (graded) Jacobson radical. In particular, the kernel of the map $A \rightarrow A'$ is generated by L_λ .

If the map $A \rightarrow A'$ is not surjective, its cokernel surjects onto some $L_\lambda \langle k \rangle$ with $k > 0$ by (4.1) and mapping cone argument. This is impossible since $[A' : L_\lambda]_A = 0$. Hence, the map $A \rightarrow A'$ must be surjective. Therefore, we conclude the first part of the assertion.

Thanks to (4.2), we have

$$\mathrm{ext}_A^\bullet(K, L_\mu) = \{0\} \quad \text{for every } \lambda \neq \mu \in \Lambda.$$

Since \widetilde{K}_λ is a projective graded A -module with simple head L_λ , we deduce the second assertion by taking the minimal projective resolution as (graded) A -modules (from Theorem 3.5). \square

Corollary 4.3. *Let $j : \mathfrak{Y} \hookrightarrow \mathfrak{X}$ be the inclusion of an open G -stable subvariety. We have*

$$\mathrm{ext}_A^*(A, L_\mu) \cong \mathrm{ext}_A^*(A_{(G, \mathfrak{Y})}, L_\mu)$$

for every $\mu \in \Lambda$ so that $\mathbb{O}_\mu \subset \mathfrak{Y}$.

Proof. By Corollary 2.8, we have

$$\mathrm{ext}_A^\bullet(\widetilde{K}_\lambda, L_\mu) = \{0\} \quad \text{for every } \lambda \prec \mu.$$

Taking account into this, a repeated application of Lemma 4.1 and Proposition 4.2 yields the result since we have $(\spadesuit)_1$. \square

Proof of Theorem 1.5. We switch to the basic ring $B_{(G, \mathfrak{X})}$ and still call it A for the sake of simplicity.

Let $\lambda \in \Lambda$ be the label of a closed G -orbit in \mathfrak{X} . We set $J := Ae_\lambda A$, where e_λ is the idempotent of A corresponding to P_λ . By Proposition 4.2, we have a short exact sequence of A -modules:

$$0 \rightarrow K \rightarrow P_\mu \rightarrow P_\mu / JP_\mu \rightarrow 0.$$

In addition, the A -module K admits a finite resolution by \tilde{K}_λ . Therefore, Theorem 1.3 implies that K is a (possibly infinite) successive extension of K_λ . Repeating this construction by using Lemma 4.1 and $(\spadesuit)_1$, we conclude that each P_μ admits a finite separable filtration whose graded quotients are (possibly infinite) extensions of $\{K_\gamma \langle i \rangle\}_{\gamma \prec \mu, i \geq 0}$.

By our rearrangement of A , each P_μ is concentrated in non-negative degrees, and each of its graded piece is finite-dimensional. In addition, each K_γ is concentrated in finitely many degrees (by its finite-dimensionality). Therefore, constructing P_μ from K_μ by applying possible extensions by $\{K_\gamma \langle i \rangle\}_{\gamma \prec \mu, i \geq 0}$ (compatible with the above filtration) yields the (quotients of the) desired filtration.

For the second assertion, observe that

$$K = \mathrm{Ext}_G^\bullet(i_* i^* \mathrm{IC}_\mu, \mathcal{L}) = \mathrm{Ext}_G^\bullet(i_! i^* \mathrm{IC}_\mu, \mathcal{L}) \cong \mathrm{Ext}_G^\bullet(i^* \mathrm{IC}_\mu, i^! \mathcal{L}),$$

where $i : \mathbb{O}_\lambda \hookrightarrow \mathfrak{X}$. Now the pointwise purity asserts that $i^! \mathcal{L}$ is a direct sum of shifted perverse sheaves (or rather local systems) by [BBD82] 5.4.5, 5.3.8. Therefore, we deduce that K is isomorphic to a finite direct sum of \tilde{K}_λ as required. \square

Proof of Theorem 1.6. Let \mathbb{O}_λ be a closed G -orbit of \mathfrak{X} . We set $\mathbb{O}_\lambda = \mathfrak{X} \setminus \mathfrak{Y}$. By Lemma 4.1, and Proposition 4.2, we have a natural short exact sequence as graded left $A_{(G, \mathfrak{X})}$ -modules

$$0 \rightarrow K \rightarrow A_{(G, \mathfrak{X})} \rightarrow A_{(G, \mathfrak{Y})} \rightarrow 0. \quad (4.3)$$

This particularly shows the purity of $A_{(G, \mathfrak{X})}$ implies that of $A_{(G, \mathfrak{Y})}$. By the second assertion of Proposition 4.2 and Theorem 1.3 4), we conclude that K is a successive extension of K_λ . Therefore, (4.3) also shows the purity of K_λ . Since we have

$$K_\lambda = H^\bullet i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda] \supset L_\lambda \boxtimes H^\bullet i_\lambda^! \mathrm{IC}_\gamma[\dim \mathbb{O}_\lambda] \quad \text{with } L_\gamma \not\cong \{0\} \text{ for every } \gamma \in \Lambda,$$

we conclude that each IC_γ has weight ≥ 0 along x_λ . Since the Verdier dual on the variety pt is the graded dual as a vector space and we have $\mathbb{D} \mathrm{IC}_\gamma \cong \mathrm{IC}_\gamma$ (as mixed sheaves, [BBD82] 5.1.8) by our choice of weight of IC_γ , we conclude that IC_γ has also weight ≤ 0 along x_λ . Hence the weight of IC_γ along x_λ is 0 for every $\gamma \in \Lambda$.

Now we apply the same argument repeatedly by replacing \mathfrak{X} with \mathfrak{Y} (and choose a closed G -orbit of \mathfrak{Y}) to proceed the induction. Since we have $(\spadesuit)_1$, the induction yields the result. \square

5 A proof of Shoji's conjecture for type B

Let $G = Sp(2n, \mathbb{C})$ be a symplectic group, and let V_1 be its vector representation. We set $V_2 := \wedge^2 V_1$ and $\mathbb{V} := V_1 \oplus V_2$. We fix a maximal torus $T \subset G$ and a Borel subgroup $B \subset G$ so that $T \subset B$. Let $W := N_G(T)/T$. Let $X^*(T)$ be the character lattice of T , which we may identify with the cocharacter lattice via a W -invariant perfect pairing. We fix a basis $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ of $X^*(T)$ so that the

set R of coroots of G and the set R^+ of positive coroots with respect to B are presented as:

$$R := \{\pm\epsilon_i \pm \epsilon_j\}_{i < j} \cup \{\pm\epsilon_i\}_{i=1}^n \supset \{\epsilon_i \pm \epsilon_j\}_{i < j} \cup \{\epsilon_i\}_{i=1}^n =: R^+.$$

For each $\beta \in X^*(T)$, let $\mathbb{V}[\beta]$ be the weight β -part of \mathbb{V} . Note that we have $\dim \mathbb{V}[\beta] \leq 1$ unless $\beta = 0$. We set $\mathbb{V}^+ := \bigoplus_{\beta \in R^+} \mathbb{V}[\beta] \subset \mathbb{V}$, which is a B -submodule. We consider a G -equivariant vector bundle $F := G \times^B \mathbb{V}^+$ and a composition map

$$\mu : F \hookrightarrow G \times^B \mathbb{V} \cong G/B \times \mathbb{V} \xrightarrow{\text{pr}_2} \mathbb{V},$$

which is again G -equivariant. We denote the image of μ by \mathfrak{N} and denote the resulting map $F \rightarrow \mathfrak{N}$ again by μ .

For each $x \in \mathfrak{N}(\mathbb{C})$, the composition map $\mu^{-1}(x) \hookrightarrow F \rightarrow G/B$ is injective. It induces a map

$$\iota_x : H_\bullet(\mu^{-1}(x), \mathbb{C}) \longrightarrow H_\bullet(G/B, \mathbb{C})$$

between the Borel-Moore homologies (see [CG97] §2 for example).

Theorem 5.1 ([K09] and Lusztig-Spaltenstein [LS79]). *We have:*

1. *The variety \mathfrak{N} has finitely many G -orbits;*
2. *For each $x \in \mathfrak{N}(\mathbb{C})$, the group $\text{Stab}_G(x)$ is connected;*
3. *The map μ is semi-small, and hence $\mu_* \mathbb{C}[\dim F]$ is a direct sum of G -equivariant simple perverse sheaves;*
4. *The sheaf $\mathcal{L} := \mu_* \mathbb{C}[\dim F]$ contains all G -equivariant simple perverse sheaves as its direct summands;*
5. *We have*

$$A := \text{Ext}_{D_G^b(\mathfrak{N})}^\bullet(\mathcal{L}, \mathcal{L}) \cong \mathbb{C}W \ltimes \mathbb{C}[\mathfrak{t}],$$

where $\mathbb{C}W$ is a group algebra of W sitting in degree 0 and $\mathbb{C}[\mathfrak{t}]$ is a polynomial algebra generated by \mathfrak{t}^ in degree 2;*

6. *The odd-part of the Borel-Moore homology $H_{\text{odd}}(\mu^{-1}(x), \mathbb{C})$ vanishes;*
7. *Let $d_x = 2 \dim \mu^{-1}(x)$. Then, $\iota_x H_{d_x}(\mu^{-1}(x), \mathbb{C})$ is an irreducible W -submodule of $H_\bullet(G/B, \mathbb{C})$ which we denote by \mathbb{L}_x ;*
8. *Let $H_{-\bullet}(G/B, \mathbb{C}) \subset \mathbb{C}[\mathfrak{t}^*]$ be the harmonic polynomial realization ([CG97] §6 with the convention $\deg \mathfrak{t} = -2$). Then, the module $\mathbb{L}_x \cong L_x \langle -d_x \rangle$ is the unique maximal degree realization (of an irreducible W -module);*
9. *There exists a reflection subgroup $W_x \subset W$ and a polynomial $p_x \in \mathbb{L}_x$ so that $\mathbb{C}p_x$ is the maximal degree realization of a one-dimensional representation of W_x inside $\mathbb{C}[\mathfrak{t}^*]$.*

Proof. The first six assertions are contained in [K09] as: The first assertion is Theorem 1.14, the second is Proposition 4.5, the third is Theorem 1.2, the fourth is Theorem 8.3, the fifth is Proposition 8.1, and the sixth is Theorem 6.2. The seventh assertion follows from [K11a] Lemma 7.6, Theorem 10.7 and [CG97]

Theorem 6.5.2 (or rather its proof). The eighth and the ninth assertions follow from Lusztig-Spaltenstein [LS79] (cf. [K11a] Theorem 10.5). Here $p_x = D(\mu, \nu)$ in [K11a], and W_x is the product of smaller Weyl groups of type BC obtained as the entries of ${}^t\mu$ and ${}^t\nu$ (see [K11a] for precise convention). \square

Since the $\mathfrak{t}^* \subset A$ in Theorem 5.1 5) is the hyperplane sections (cf. [K09] §8), we deduce that each ι_x is an A -module map.

Proposition 5.2. *In the notation of section one, we have: The G -variety \mathfrak{N} satisfies the conditions (\spadesuit) and (\clubsuit) . In addition, we have $A = A_{(G, \mathfrak{N})}$.*

Proof. Theorem 5.1 1) and 2) asserts (\spadesuit) , 3) and 4) imply that the algebra A (in Theorem 5.1 5)) is the one we described in section one, and 6) yields $(\clubsuit)_1$.

We show $(\clubsuit)_2$. By [K09] Corollary 4.7, we deduce that the reductive part of the group $\text{Stab}_G(x_\lambda)$ is a product of symplectic groups, say $Sp(2n_1) \times \cdots \times Sp(2n_k)$. In addition, each of these symplectic groups preserves blocks (cf. [K09] Definition 1.11 with $\ell = 0$) of the same length, say j_1, \dots, j_k , respectively. It follows that (a suitable choice of) the reductive part of the embedding $\text{Stab}_G(x_\lambda) \subset Sp(2n)$ factors through

$$Sp(2n_1) \times \cdots \times Sp(2n_k) \hookrightarrow Sp(2n_1)^{j_1} \times \cdots \times Sp(2n_k)^{j_k} \hookrightarrow Sp(2n),$$

where the first embedding is the product of diagonal embeddings $Sp(2n_i) \subset Sp(2n_i)^{j_i}$ and the second embedding is diagonal (in the sense the restriction of T yields the maximal torus; we have $\sum_i n_i j_i \leq n$). Therefore, we have the corresponding embeddings of maximal tori

$$T_1 \times \cdots \times T_k \hookrightarrow T_1^{j_1} \times \cdots \times T_k^{j_k} \hookrightarrow T.$$

Since j_1, j_2, \dots, j_k are all distinct, we deduce that

(\star) a lift $\dot{w} \in N_G(T)$ of $w \in W$ preserves $T_1 \times \cdots \times T_k$ if and only if $\text{Ad} \dot{w}$ preserves each T_i .

Here we have

$$H_{\text{Stab}_G(x_\lambda)}^\bullet(\text{pt}) \cong \bigotimes_{i=1}^k H_{\text{Sp}(2n_i)}^\bullet(\text{pt}) \cong \bigotimes_{i=1}^k H_{T_i}^\bullet(\text{pt})^{W_i},$$

where $W_i := N_{\text{Sp}(2n_i)}(T_i)/T_i$. Therefore, (\star) is enough to deduce that the restriction map $H_T^\bullet(\text{pt}) \rightarrow \bigotimes_{i=1}^k H_{T_i}^\bullet(\text{pt})$ yields the surjective map

$$H_G^\bullet(\text{pt}) = H_T^\bullet(\text{pt})^W \twoheadrightarrow \bigotimes_{i=1}^k H_{T_i}^\bullet(\text{pt})^{W_i} = H_{\text{Stab}_G(x_\lambda)}^\bullet(\text{pt}).$$

This is $(\clubsuit)_2$. \square

Thanks to Proposition 5.2, we can apply the machinery and notation of section one. In particular, we have $\Lambda = G \backslash \mathfrak{N}$ and its closure relation \prec .

We set $A_n := A$ and replace x with the label $\lambda \in \Lambda$ so that $x \in \mathbb{O}_\lambda(\mathbb{C})$ in the below. For each $\lambda \in \Lambda$ and $x_\lambda \in \mathbb{O}_\lambda(\mathbb{C})$, we have

$$\begin{aligned} K_\lambda &= H^\bullet i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda] \cong \bigoplus_{i \in \mathbb{Z}} H^i i_\lambda^! \mathcal{L}[\dim \mathbb{O}_\lambda] \\ &= \bigoplus_{i \in \mathbb{Z}} H^i i_\lambda^! \mu_* \mathbb{C}[\dim F + \dim \mathbb{O}_\lambda] \cong \bigoplus_{i \in \mathbb{Z}} H^i(\mu^{-1}(x_\lambda), \omega[-d_\lambda]), \end{aligned}$$

where ω is the dualizing complex of $\mu^{-1}(x_\lambda)$ and the last equation follows from the base change (and semi-smallness for the grading). This implies that $K_\lambda \cong H_{-\bullet}(\mu^{-1}(x_\lambda)) \langle d_\lambda \rangle$ as graded A -modules.

Recall that each K_λ is a graded A -module presented as:

$$K_\lambda = P_\lambda / \sum_{f \in \text{hom}_A(P_\mu, P_\lambda)^{>0}, \mu \preceq \lambda} \text{Im } f$$

by Theorem 1.3 3) and 4). In other words, K_λ is a \mathcal{P} -trace in the terminology of [K11b].

Theorem 5.3 (Shoji's conjecture [Sho04] 3.13). *For each distinct $\lambda, \mu \in \Lambda$, we have*

$$\langle K_\lambda, K_\mu^* \rangle_{\text{gEP}} = 0. \quad (5.1)$$

In addition, each ι_λ is an inclusion of A -modules.

Remark 5.4. Thanks to [AH08] §5 and [K11b] 2.17, Theorem 5.3 verifies [AH08] Conjecture 6.4 (2) since the both sides obey the same Lusztig-Shoji algorithm up to substitution $t \mapsto t^2$.

Proof of Theorem 5.3. The original conjecture of Shoji [Sho04] 3.13 is stated as a statement on the Lusztig-Shoji algorithm on limit symbols and its relation with coinvariants. The equivalence of the Lusztig-Shoji algorithm [Sho83, Sho88] with (5.1) is [K11b] Theorem 2.17. The compatibility of the orders arising from limit symbols and orbit closures of \mathfrak{N} is proved in Achar-Henderson [AH08] Theorem 3.9 and Theorem 6.1. Therefore, we apply Corollary 3.10 to deduce the first part of the assertion.

We prove the second assertion. Let $M_\lambda := \iota_\lambda K_\lambda \langle -d_\lambda \rangle$. Since K_λ has simple head as a graded A -module, the A -module M_λ is obtained by the A -module saturation of L_λ (where the natural $\mathbb{C}[t]$ -action is via the derivations). By the \mathcal{P} -trace presentation of K_λ and Theorem 1.3 1), the injectivity of ι_λ is equivalent to

$$\text{ext}_A^1(M_\lambda, L_\mu) = \{0\} \quad \text{for every } \lambda \prec \mu. \quad (5.2)$$

If $W_\lambda = W$, then we have $L_\lambda \cong \text{sgn}$ or Ssgn by [K11b] Fact 4.1. We prove the assertion (5.2) in the both case. These cases correspond to **a)** $x_\lambda = 0$, and **b)** $\mathbb{O}_\lambda = (V_1 \setminus \{0\}) \oplus \{0\} \subset \mathbb{V}$, respectively. For the case **a)**, the assertion $M_\lambda = K_\lambda \langle -d_\lambda \rangle$ is standard since the both sides are $(\mathbb{C}[t] / \langle \mathbb{C}[t]_+^W \rangle)^*$ and $H_{-\bullet}(G/B)$, respectively (cf. [CG97] §6.4). For the case **b)**, the (G -equivariant) composition map $F \rightarrow \mathfrak{N} \rightarrow V_1$ is surjective, and is regular along x_λ . It follows that $\mu^{-1}(x_\lambda)$ is a smooth projective variety. In particular, we have the Poincaré duality $K_\lambda^* \cong K_\lambda \langle -d_\lambda \rangle$, which intertwines the action of the hyperplane sections. Since $\dim L_\lambda = 1$, the module K_λ has simple head as a $\mathbb{C}[t]$ -module. Therefore, we conclude that $K_\lambda \langle -d_\lambda \rangle$ must have simple socle of degree 0. By construction, we also have $M_\lambda^0 \neq \{0\}$. This forces $K_\lambda \langle -d_\lambda \rangle \cong M_\lambda$.

In the below, we examine the case $W_\lambda \neq W$. If we have $\lambda \prec \mu$, then we have $\mathbb{C}p_\lambda \not\subset L_\mu$ as W_λ -modules by $d_\lambda > d_\mu$ and Theorem 5.1 9).

Let $A_\lambda := \mathbb{C}W_\lambda \ltimes \mathbb{C}[t] \subset A$. We have $A_\lambda \neq A$ by $W_\lambda \neq W$. We define M_λ^\downarrow as the A_λ -submodule of M_λ generated by $\mathbb{C}p_\lambda$. We have

$$\mathbb{C}WM_\lambda^\downarrow = M_\lambda \quad (5.3)$$

by $\mathbb{C}W A_\lambda = A$. Since the both of $\mathbb{C}W$ and $\mathbb{C}W_\lambda$ are semisimple algebras, an element of $\text{ext}_A^1(M_\lambda, L_\mu)$ induce an extension as A_λ -modules. In particular, the non-vanishing of the LHS of (5.2) implies

$$\text{ext}_{A_\lambda}^1(M_\lambda^\downarrow, L_\mu) \neq \{0\} \quad \text{for some } \lambda \prec \mu$$

from (5.3). By induction on n , we deduce that $M_\lambda^\downarrow \cong \boxtimes_{i=1}^m K_{\lambda_i}$ if $A_\lambda \cong \boxtimes_{i=1}^m A_{n_i}$ with $\sum_{i=1}^m n_i = n$. By Corollary 2.8 2), each direct summand of the minimal projective resolution of M_λ^\downarrow as a graded A_λ -module is of the form $\boxtimes_{i=1}^m P_{\gamma_i}$ with $\gamma_i \preceq \lambda_i$ for every $i = 1, \dots, m$ up to a grading shift. It follows that

$$\text{hom}_{W_\lambda}(\boxtimes_{i=1}^m L_{\gamma_i}, L_\mu) \neq \{0\}$$

for some $\lambda \prec \mu$ if (5.2) fails. By Theorem 5.1 9), the W_λ -module $\boxtimes_{i=1}^m L_{\gamma_i}$ is realized inside $\mathbb{C}[\mathfrak{t}^*]$ only at degree

$$-\sum_{i=1}^m d_{\gamma_i} \leq -\sum_{i=1}^m d_{\lambda_i} = -d_\lambda < -d_\mu.$$

Therefore, we deduce $\text{ext}_A^1(M_\lambda, L_\mu) = \{0\}$ also in the case $W_\lambda \neq W$, and this finishes the proof. \square

Corollary 5.5 (Achar-Henderson [AH08] Conjecture 6.4 (1)). *For each $\lambda, \mu \in \Lambda$, we have $[K_\lambda : L_\mu] = t^k Q(t^4)$ for some $k \geq 0$ and $Q \in \mathbb{N}[t]$.*

Proof. By [K11b] Fact 4.1 (1) and (6), if L_λ corresponds to a bi-partition $(\lambda^{(0)}, \lambda^{(1)})$, then each $L_\mu \subset (\mathfrak{t}^*)^{\otimes l} \otimes_{\mathbb{C}} L_\lambda$ ($l \geq 0$) corresponds to a bi-partition $(\mu^{(0)}, \mu^{(1)})$ with $|\mu^{(0)}| - |\lambda^{(0)}| \equiv l \pmod{2}$. Since $P_\lambda \cong \mathbb{C}[\mathfrak{t}] \otimes L_\lambda$ as graded W -modules (cf. [K11b] §1), we deduce $[P_\lambda : L_\mu] = t^{k'} Q'(t^4)$ for some $k' \geq 0$ and $Q' \in \mathbb{N}[[t]]$. As K_λ is a quotient of P_λ , we conclude the result. \square

Corollary 5.6. *For each $x \in \mathfrak{N}(\mathbb{C})$, the variety $\mu^{-1}(x)$ has a pure homology.*

Proof. The existence of an affine paving of the variety $F \times_{\mathfrak{N}} F$ can be read off from [K09] Lemma 1.5. This implies the purity of A by [CG97] §8. Therefore, we apply Theorem 1.6 to Proposition 5.2 to conclude that each G -equivariant simple perverse sheaf of weight 0 is pointwise pure of weight 0. The result now follows by the definition of K_λ . \square

Appendix: Coinvariants and Lieb-McGuire systems

We work in the setting of section five. We set $\alpha_i^\vee := \epsilon_i - \epsilon_{i+1}$ for each $1 \leq i \leq n$ (with $\epsilon_{n+1} := 0$). We fix two parameters $m, r \in \mathbb{R}$. Let $\mathcal{H}_{r,m}$ be the algebra which contains the group ring $\mathbb{C}W$ and the polynomial ring $\mathbb{C}[\mathfrak{t}] = \mathbb{C}[\epsilon_1, \dots, \epsilon_n]$ with the following properties:

1. $\mathcal{H}_{r,m} \cong \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{t}]$ as vector spaces;
2. For each $1 \leq i \leq n$ and each $f \in \mathbb{C}[\mathfrak{t}]$, we have

$$s_i \cdot f - s_i f \cdot s_i = \begin{cases} r \frac{f - s_i f}{\alpha_i^\vee} & (i \neq n) \\ mr \frac{f - s_n f}{\alpha_n^\vee} & (i = n) \end{cases},$$

where ${}^w f$ ($w \in W$) denotes the natural w -action on $\mathbb{C}[\mathfrak{t}]$.

Let \mathbf{e} be the idempotent of $\mathbb{C}W$ corresponding to the trivial representation. We have an isomorphism $\mathbb{C}[\mathfrak{t}] \ni f \mapsto f\mathbf{e} \in \mathcal{H}_{r,m}\mathbf{e}$, by which we regard $\mathbb{C}[\mathfrak{t}]$ as a representation of $\mathcal{H}_{r,m}$.

We define an anti-isomorphism $\dagger : \mathcal{H}_{r,m} \rightarrow \mathcal{H}_{-r,m}$ as:

$$s_i^\dagger := s_i, \text{ and } \epsilon_j^\dagger = -\epsilon_j \text{ for every } 1 \leq i, j \leq n.$$

The center of $\mathcal{H}_{r,m}$ is isomorphic to $\mathbb{C}[\mathfrak{t}]^W$ via the natural inclusion (see [Lus89]), and hence each W -orbit $W\gamma$ (which we may simply denote by γ) of \mathfrak{t} defines a central character $\mathbb{C}[\mathfrak{t}]^W \rightarrow \mathbb{C}$ ($=: \mathbb{C}_\gamma$). Let $\text{Irr}_\gamma \mathcal{H}_{r,m}$ be the set of isomorphism classes of irreducible $\mathcal{H}_{r,m}$ -modules with a central character γ .

Theorem A.1 (Heckman-Opdam [HO97]). *Let \mathcal{R} be the space of C^∞ -functions with respect to ξ_1, \dots, ξ_n for which ϵ_i acts by $\frac{\partial}{\partial \xi_i}$. We set $\mathbb{C}[\mathfrak{t}^*] := \mathbb{C}[\xi_1, \dots, \xi_n] \subset \mathcal{R}$.*

1. *There exists a $\mathcal{H}_{-r,m}$ -action on \mathcal{R} so that the pairing*

$$\mathbb{C}[\mathfrak{t}] \times \mathcal{R} \ni (P, f) \mapsto (Pf)(0) \in \mathbb{C} \tag{A.4}$$

interchanges the $\mathcal{H}_{r,m}$ -module structures on $\mathbb{C}[\mathfrak{t}]$ with the $\mathcal{H}_{-r,m}$ -module structures of $\mathbb{C}[\mathfrak{t}^]$ and \mathcal{R} as:*

$$(TP, f) = (P, T^\dagger f) \text{ for every } T \in \mathcal{H}_{r,m}, P \in \mathbb{C}[\mathfrak{t}], \text{ and } f \in \mathcal{R};$$

2. *For each $\gamma \in \mathfrak{t}$, there exists a unique non-zero function $\phi_\gamma \in \mathcal{R}$ up to scalar such that:*

$$\mathbb{C}[\mathfrak{t}]^W \phi_\gamma \cong \mathbb{C}_\gamma \text{ as } \mathbb{C}[\mathfrak{t}]^W\text{-modules; and } W \text{ acts on } \mathbb{C}\phi_\gamma \text{ by triv;}$$

3. *If we set $M_{\gamma,m} := \mathcal{H}_{-r,m}\phi_\gamma$, then it is irreducible as a $\mathcal{H}_{-r,m}$ -module.*

The algebra $\mathcal{H}_{r,m}$ specializes to A (in §5) by setting $r = 0$.

We put $\mathbb{G} = G \times (\mathbb{C}^\times)^2$. We consider the following condition (\star) on $a = (s, \vec{q}) \in \mathbb{G}$:

$$(\star)_0 \quad \vec{q} = (q^m, q), \text{ where } q = e^r \in \mathbb{R}_{>1} \text{ and } m \in \mathbb{R};$$

$$(\star)_1 \quad s = \exp(\gamma) \text{ with } \gamma \in \mathfrak{t}(\mathbb{R}).$$

Theorem A.2 (Standard modules, [K09] §7, §9). *Assume that $a \in \mathbb{G}$ satisfies (\star) . Let $v \in \mathbb{V}$ so that $av = v$. We have a $\mathcal{H}_{-r,m}$ -module*

$$M_{(a,v)} := H_\bullet(\mathcal{E}_v^a),$$

which is isomorphic to K_λ for some $\lambda \in \Lambda$ as W -modules.

Theorem A.3 (eDL correspondence, [K09], §10). *Assume that $a \in \mathbb{G}$ satisfies (\star) . Then, we have a one-to-one correspondence*

$$\text{Irr}_\gamma \mathcal{H}_{-r,m} \ni L_{(a,v)} \longleftrightarrow v \in Z_G(s) \backslash \mathbb{V}^a.$$

Moreover, $L_{(a,v)}$ is a quotient of $M_{(a,v)}$ as a $\mathcal{H}_{-r,m}$ -module.

Lemma A.4. *Let $L \subset \mathbb{C}[\mathfrak{t}^*]$ be a homogeneous W -submodule isomorphic to L_λ for some $\lambda \in \Lambda$. Then, we have*

$$K_\lambda \langle -d_\lambda \rangle \subset AL.$$

Proof. By Theorem 5.1 9), $\mathbb{C}p_\lambda \subset K_\lambda \langle -d_\lambda \rangle \subset \mathbb{C}[\mathfrak{t}^*]$ is a one-dimensional representation of W_λ which is its maximal degree realization. Let $\mathbb{C}[\mathfrak{t}^*]^{W_\lambda}$ denote the W_λ -invariant part of $\mathbb{C}[\mathfrak{t}^*]$.

There exists $q \in L$ for which $\mathbb{C}W_\lambda q \cong \mathbb{C}W_\lambda p_\lambda$ as (one-dimensional) W_λ -modules. It follows that we have a factorization

$$q = p_\lambda \cdot r \quad \text{with} \quad r \in \mathbb{C}[\mathfrak{t}^*]^{W_\lambda}.$$

There exists $Q^+ \in \mathbb{C}[\mathfrak{t}]$ so that $Q^+ q = 1 \in \mathbb{C}[\mathfrak{t}]$ by the non-degeneracy of the pairing (A.4) restricted to $\mathbb{C}[\mathfrak{t}^*] \subset \mathcal{R}$. Here we can rearrange Q^+ if necessary to assume that $\mathbb{C}Q^+$ is isomorphic to $\mathbb{C}p_\lambda$ as a W_λ -module. It follows that Q^+ admits a factorization $Q^+ = PQ$, where $P \in \mathbb{C}[\mathfrak{t}]$ is the minimal degree realization of $\mathbb{C}p_\lambda$ inside non-negatively graded ring $\mathbb{C}[\mathfrak{t}]$, and Q is W_λ -invariant. By the comparison of degrees, we conclude that $0 \neq Qq \in \mathbb{C}p_\lambda$, which implies the result. \square

Theorem A.5. *Let $\gamma \in \mathfrak{t}$. Then, there exists $\lambda \in \Lambda$ so that the vanishing order of $M_{m,\gamma}$ along 0 induces a graded W -module structure equal to K_λ .*

Remark A.6. In Theorem A.1, the function ϕ_γ gives rise to a unique (up to scalar) solution of the Lieb-McGuire system so that $\mathbb{C}[\mathfrak{h}]^W$ acts by γ along the set of regular points of \mathfrak{h} (see [HO97] for detail). In this sense, the vanishing order filtration of $M_{m,\gamma}$ measures the structure of the Taylor series of ϕ_γ .

Proof of Theorem A.5. For each $f \in \mathcal{R}$, let $\text{lt } f$ denote the maximal degree non-zero homogeneous component of the Taylor expansion of f along 0 (remember that our degree counting on \mathcal{R} is nonpositive). Then, the vanishing-order filtration of $M_{m,\gamma}$ is transformed into the graded structure of the module

$$\text{gr } M_{m,\gamma} := \{\text{lt } f \mid f \in M_{m,\gamma}\}.$$

Since $M_{m,\gamma}$ is $\mathbb{C}[\mathfrak{t}]$ -stable, so is the space $\text{gr } M_{m,\gamma}$. Here [HO97] formula (2.1) implies that $s_i \in W$ acts on $\mathbb{C}[\mathfrak{t}^*]$ by letting s_i acts by the natural homogeneous action and adds some lower order terms. It follows that $\text{gr } M_{m,\gamma} \subset \mathbb{C}[\mathfrak{t}^*]$ is an A -submodule. Here $M_{m,\gamma}$ contains triv as a W -module. It implies that the irreducible module $M_{m,\gamma}$ is isomorphic to some standard module in the sense of Theorem A.2 (cf. [CK11] Corollary 1.20). Hence, Theorem A.3 asserts that $\text{gr } M_{m,\gamma} \cong K_\lambda$ as a W -module for some $\lambda \in \Lambda$. By Theorem 5.1 8), we deduce that the image of the natural composition map

$$\varphi : L_\lambda \hookrightarrow \text{gr } M_{m,\gamma} \subset \mathbb{C}[\mathfrak{t}^*]$$

lands in degree $\leq -d_\lambda$. Since $\dim \text{Hom}_W(L_\lambda, M_{m,\gamma}) = 1$, we conclude that $\text{Im } \varphi$ is homogeneous. If $\text{Im } \varphi = L_\lambda$ (i.e. the degree is $-d_\lambda$ by Theorem 5.1 8)), then we have $\text{gr } M_{m,\gamma} = K_\lambda \langle -d_\lambda \rangle$ by the inclusion as A -modules and the comparison of dimensions.

If $\text{Im } \varphi$ is of degree $< -d_\lambda$, then Lemma A.4 implies that $\dim \text{Hom}_W(L_\lambda, M_{m,\gamma}) \geq 2$. Hence, this case cannot occur as desired. \square

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